7.2 Equivalent Vector Channel

7.21. Recall that we are considering the digital modulator/demodulator part shown in Figure 37.

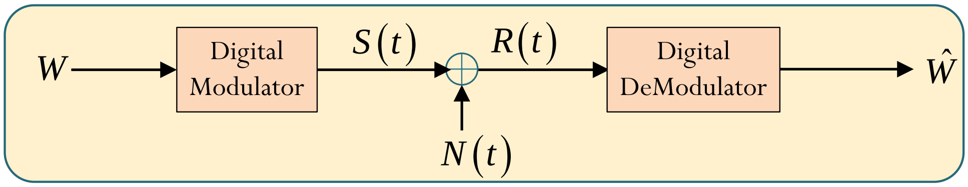

Figure 37: Digital modulator/demodulator and the waveform channel

7.22. The input of the modulator is the (random) message (index) $W \in \{1, 2, \ldots M\}$.

- Prior probabilities: $p_j = P[W = j]$.
- Each message is mapped to a waveform to be transmitted over the waveform channel as the transmitted waveform $S(t)$.
    - There are M possible messages. So, there are M waveforms:

    $$s_1(t), s_2(t), \ldots, s_M(t).$$

    The (symbol) energy of the j-th waveform is $E_j = \langle s_j(t), s_j(t) \rangle$. The average energy per symbol is $E_s = \sum_{j=1}^{M} p_j E_j$.
    - Transmission of the message $W = j$ is done by inputting the corresponding waveform $s_j(t)$ into the channel.
    Therefore, the probability that the waveform $s_j(t)$ is selected to be transmitted is the same as the probability that the j^{th} message occurs:

    $$p_j = P[W = j] = P[S(t) = s_j(t)]$$

7.23. The noise $N(t)$ in the channel is assumed to be 1 additive. So, the receiver observes $R(t) = S(t) + N(t)$. The noise is also assumed to be 2 independent from the transmitted waveform $S(t)$.

7.24. Conversion of Waveform Channels to Vector Channels:

(a) Given M waveforms $s_1(t), s_2(t), \ldots, s_M(t)$, first find (possibly by GSOP) the K orthonormal basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_K(t)$ for the space spanned by $s_1(t), s_2(t), \ldots, s_M(t)$.

(b) The basis gives the vector representations $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \ldots, \mathbf{s}^{(M)}$ for the waveforms $s_1(t), s_2(t), \ldots, s_M(t)$, respectively. Note that $s_i^{(j)}$, the i^{th} component of the vector $\mathbf{s}^{(j)}$, comes from the inner-product:

$$s_i^{(j)} = \langle s_j(t), \phi_i(t) \rangle .$$

(c) The vector representations of the received waveform and the noise can then be calculated in a similar manner based on the derived basis.

(d) In summary, we convert the waveforms $S(t)$, $R(t)$, and $N(t)$ to their corresponding vectors $\mathbf{S}$, $\mathbf{R}$, and $\mathbf{N}$ by performing inner-product with the orthonormal axes: the i-th component of the vector is the inner-product between the waveform and $\phi_i(t)$. In particular,

$$S_i = \langle S(t), \phi_i(t) \rangle, \quad R_i = \langle R(t), \phi_i(t) \rangle, \quad N_i = \langle N(t), \phi_i(t) \rangle .$$

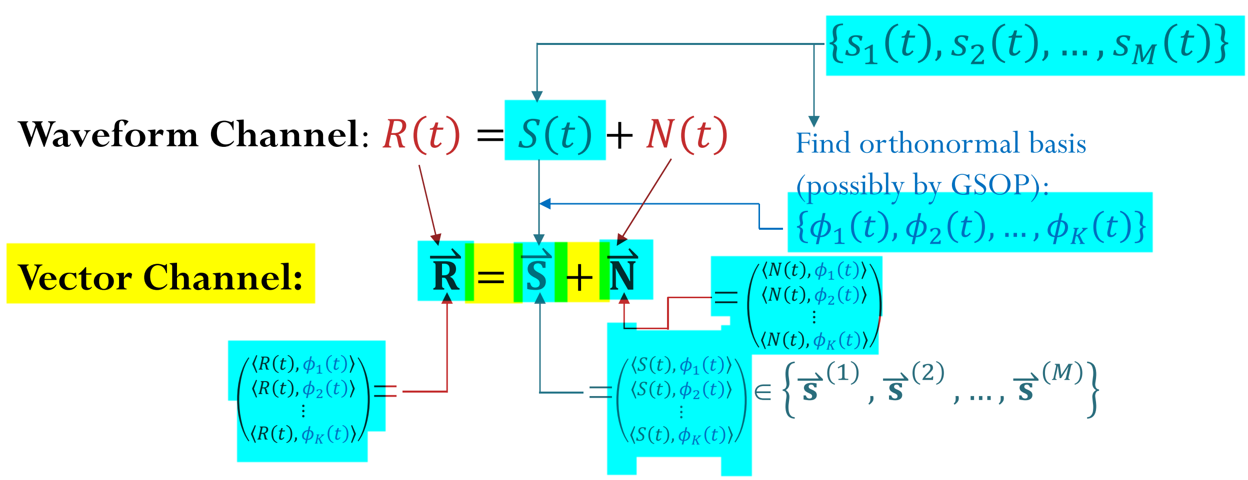

Figure 38: Conversion of Waveform Channels to Vector Channels

Remarks:

- We use the letter K instead of the letter N to represent the number of orthonormal basis functions to avoid the confusion with the random noise which is also denoted by the letter N.

- This conversion is the same as what we did when we convert waveforms to vectors via the GSOP. (See Eq. (38) and Figure 27a.) When $s_j(t)$ is transmitted, the corresponding "transmitted" vector will be $\mathbf{s}^{(j)}$.

Example 7.25. Figure 39 illustrates how the message vectors in quaternary QAM are corrupted by additive noise.

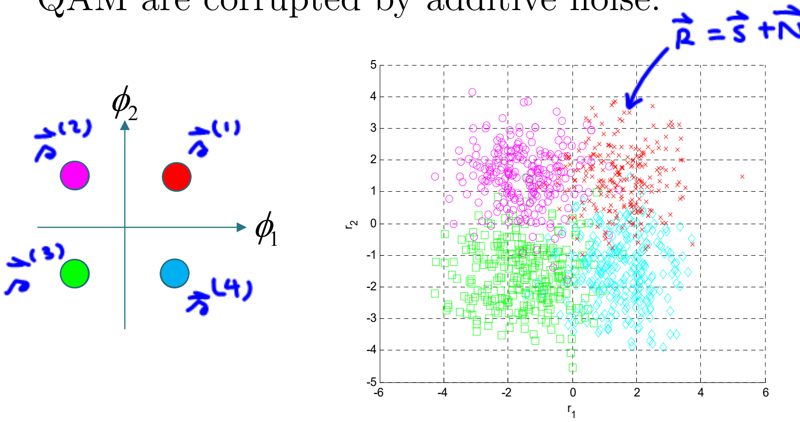

Figure 39: Quaternary QAM: constellation and samples of the received vectors (which are corrupted by additive noise)

7.26. Some facts that followed from the conversion:

(a) For $R(t) = S(t) + N(t)$, we have $\mathbf{R} = \mathbf{S} + \mathbf{N}$.

(b) From the perspective of designing optimal demodulator, the waveform channel and the vector channel are "equivalent".

(c) $E_j = \langle s_j(t), s_j(t) \rangle = \left\langle \mathbf{s}^{(j)}, \mathbf{s}^{(j)} \right\rangle$.

$\langle r(t), s_j(t) \rangle = \langle \vec{r}, \vec{s}^{(j)} \rangle$

(d) Prior probabilities:

$$p_j = P[W = j] = P[S(t) = s_j(t)] = P\left[\mathbf{S} = \mathbf{s}^{(j)}\right]$$

(e) $\mathbf{S} \perp\!\!\!\perp \mathbf{N}$

(f) When $N(t)$ is a white noise process with $S_N(f) \equiv \frac{N_0}{2}$ (across all frequencies under consideration, we have

$N(t) \longrightarrow \vec{N} = \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_K \end{pmatrix}$

(i) $\mathbb{E}[N_j] = 0$, and

(ii) $\mathbb{E}[N_i N_j] = \begin{cases} N_0/2, & i = j, \\ 0, & i \neq j. \end{cases}$

In other words, the noise components are uncorrelated and

$$\mathbb{E}\left[N_i^2\right] = \operatorname{Var} N_i = \frac{N_0}{2}.$$

$\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$

$\sigma_{N_i} = \sqrt{\operatorname{Var} N_i} = \sqrt{\frac{N_0}{2}}$